

## SLICING MIXED BING–WHITEHEAD DOUBLES

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ABSTRACT. We show that if  $K$  is any knot whose Ozsváth–Szabó concordance invariant  $\tau(K)$  is positive, the all-positive Whitehead double of any iterated Bing double of  $K$  is topologically but not smoothly slice. We also show that the all-positive Whitehead double of any iterated Bing double of the Hopf link (e.g., the all-positive Whitehead double of the Borromean rings) is not smoothly slice; it is not known whether these links are topologically slice.

## 1. INTRODUCTION

A knot in the 3-sphere is called *topologically slice* if it bounds a locally flatly embedded disk in the 4-ball, and *smoothly slice* if the disk can be taken to be smoothly embedded. Two knots are called (topologically or smoothly) *concordant* if they are the ends of an embedded annulus in  $S^3 \times I$ ; thus, a knot is slice if and only if it is concordant to the unknot. More generally, a link is (topologically or smoothly) *slice* if it bounds a disjoint union of appropriately embedded disks. The study of concordance — especially regarding the relationship between the notions of topological and smooth sliceness — is one of the major areas of active research in knot theory, and it is closely tied to the perplexing differences between topological and smooth 4-manifold theory.

Given a knot  $K \subset S^3$ , the (*untwisted*) *positive and negative Whitehead doubles* of  $K$ ,  $Wh_+(K)$  and  $Wh_-(K)$ , and the *Bing double* of  $K$ ,  $B(K)$ , are the satellites of  $K$  illustrated in Figure 1 (for  $K$  the figure-eight knot). The Whitehead doubles of a link  $L$  are obtained by doubling the individual components of  $L$ , with a choice of sign for each component. In particular, we denote the all-positive and all-negative Whitehead doubles of  $L$  by  $Wh_+(L)$  and  $Wh_-(L)$ , respectively.

Whitehead and Bing doubling play a central role in the study of concordance. In the topological setting, Freedman [7, 9] proved that any Whitehead double of a knot or, more generally, a *boundary link* (a link whose components bound disjoint Seifert surfaces) is topologically slice. Moreover, the surgery conjecture for 4-manifolds with arbitrary fundamental group — the central open problem in four-dimensional topology — is equivalent to the conjecture that the Whitehead double of any link whose linking numbers are all zero is freely topologically slice.<sup>1</sup> This conjecture is true for two-component links [8] but open in general. To disprove the surgery conjecture for manifolds with free fundamental group, it would thus suffice to show that one such

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<sup>1</sup>A link  $L$  is *freely slice* if it bounds slice disks in  $B^4$  whose complement has free fundamental group.

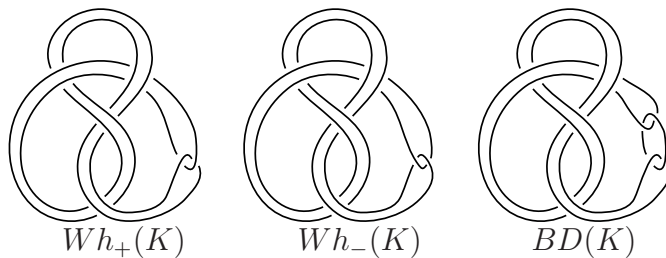


FIGURE 1. The positive and negative Whitehead doubles and the Bing double of the figure-eight knot.

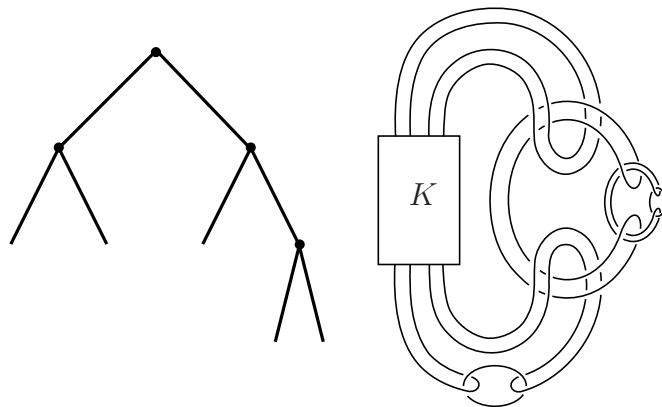


FIGURE 2. A binary tree  $T$  and the corresponding iterated Bing double  $B_T(K)$ .

link — e.g., a Whitehead double of the Borromean rings — is not (freely) topologically slice. However, all such links have resisted all attempts to determine whether or not they are topologically slice.

Around the same time, the advent of Donaldson's gauge theory made it possible to show that some of Freedman's examples of topologically slice knots are not smoothly slice. Akbulut [unpublished] first proved in 1983 that the positive, untwisted Whitehead double of the right-handed trefoil is not smoothly slice. Later, using results of Kronheimer and Mrowka on Seiberg–Witten theory, Rudolph [28] showed that any nontrivial knot that is *strongly quasipositive* cannot be smoothly slice. In particular, the positive, untwisted Whitehead double of a strongly quasipositive knot is strongly quasipositive; thus, by induction, any iterated positive Whitehead double of a strongly quasipositive knot is topologically but not smoothly slice. Biřaca [1] used this result to give explicit constructions of exotic smooth structures on  $\mathbb{R}^4$ . Later, Hedden [12] generalized this result to any knot  $K$  whose Ozsváth–Szabó invariant  $\tau(K)$  (an integer-valued concordance invariant coming from the knot Floer homology of  $K$  [21, 26]) is positive. It is conjectured [14, Problem 1.38] that if  $Wh_{\pm}(K)$  is smoothly slice, then  $K$  itself must also be smoothly slice.

We may consider *partially iterated Bing doubles* of any link: at each stage in the iteration, we replace some component by its Bing double. Specifically, given a knot  $K$ , a binary tree  $T$  specifies such a link  $B_T(K)$ , as illustrated in Figure 2, with one component for each leaf of  $T$ . For a link  $L = K_1 \cup \cdots \cup K_n$  and binary trees  $T_1, \dots, T_n$ , we may similarly obtain a link  $B_{T_1, \dots, T_n}(L) = B_{T_1}(K_1) \cup \cdots \cup B_{T_n}(K_n)$ . In particular, if  $H$  is the Hopf link, the links obtained in this manner are known as *generalized Borromean links*, since Bing doubling one component of  $H$  yields the Borromean rings.

Using the author’s work in [16] — a lengthy computation of  $\tau$  for a particular family of satellite knots — we shall prove:

**Theorem 1.1.** (1) *Let  $K$  be a knot with  $\tau(K) > 0$ , and let  $T$  be any binary tree. Then the all-positive Whitehead double of  $B_T(K)$ ,  $Wh_+(B_T(K))$ , is not smoothly slice.*

(2) *Let  $H = K_1 \cup K_2$  denote the Hopf link, and let  $T_1, T_2$  be binary trees. Then  $Wh_+(B_{T_1, T_2}(H))$  is not smoothly slice.*

Note that for any knot  $K$ ,  $B_T(K)$  is always a boundary link (see [4] for a proof), so any Whitehead double of  $B_T(K)$  (with clasps of either sign) is topologically slice. Thus, part (1) of Theorem 1.1 provides a large family of links that are topologically but not smoothly slice.<sup>2</sup> On the other hand, it is unknown whether the links in part (2) — the all-positive Whitehead doubles of the generalized Borromean links — are topologically slice. Indeed, Freedman [9] showed that the family of Whitehead doubles (with any signs) of generalized Borromean links is “atomic” for the surgery problem: whether or not they are freely topologically slice is equivalent to the surgery conjecture. Most experts nowadays conjecture that these links are *not* topologically slice, but the problem remains unsolved after nearly twenty-five years.

The requirement that we consider all-positive Whitehead doubles is necessary for our proof of Theorem 1.1. By taking mirrors, we also see that the all-negative Whitehead doubles of iterated Bing doubles of knots with  $\tau(K) < 0$  or of generalized Borromean links are not smoothly slice, but our method always fails when both positive and negative Whitehead doubling are used. Indeed, all of the gauge-theoretic invariants known to date suffer from the same asymmetry. It is still not known whether, for instance, the positive untwisted Whitehead double of the left-handed trefoil is smoothly slice.

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<sup>2</sup>The question of when iterated Bing doubles of a knot are slice is also quite challenging, since the classical sliceness obstructions vanish for iterated Bing doubles. Recent papers by Cimasoni [4] and Cha–Livingston–Ruberman [3], Cha–Kim [2], and Van Cott [30] show that if an iterated Bing double of  $K$  is topologically slice, then  $K$  is algebraically slice; if it is smoothly slice, then  $\tau(K) = 0$ . Also, Cochran, Harvey, and Leidy [5] have used  $L^2$  signatures to find algebraically slice knots with non-slice iterated Bing doubles.

Petkova, and Jen Hom for many helpful conversations about bordered Heegaard Floer homology; and to Rob Schneiderman, Charles Livingston, and Matthew Hedden for their suggestions regarding link concordance questions.

## 2. DEFINITIONS

We begin by giving more precise definitions of some of the terms used in the Introduction.

**2.1. Infection and doubling operators.** We always work with *oriented* knots and links. For any knot  $K \subset S^3$ , let  $K^r$  denote  $K$  with reversed orientation, let  $\bar{K}$  denote the mirror of  $K$  (the image of  $K$  under a reflection of  $S^3$ ), and let  $-K = \bar{K}^r$ . As  $K \# -K$  is always smoothly slice, the concordance classes of  $K$  and  $-K$  are inverses in  $\mathcal{C}_1$ , which justifies this choice of notation. Note that the invariants coming from Heegaard Floer homology ( $\widehat{\text{HFK}}(S^3, K)$ ,  $\tau(K)$ , etc.) are sensitive to mirroring but not to reversing the orientation of a knot.

Suppose  $L$  is a link in  $S^3$ , and  $\gamma$  is an oriented curve in  $S^3 \setminus L$  that is unknotted in  $S^3$ . For any knot  $K \subset S^3$  and  $t \in \mathbb{Z}$ , we may form a new link  $I_{\gamma, K, t}(L)$ , the *t-twisted infection of  $L$  by  $K$  along  $\gamma$* , by deleting a neighborhood of  $\gamma$  and gluing in a copy of the exterior of  $K$  by a map that takes a Seifert-framed longitude of  $K$  to a meridian of  $\gamma$  and a meridian of  $K$  to a  $t$ -framed longitude of  $\gamma$ . Since  $S^3 \setminus \gamma = S^1 \times D^2$ , the resulting 3-manifold is simply  $\infty$  surgery on  $K$ , i.e.  $S^3$ ; the new link  $I_{\gamma, K, t}(L)$  is defined as the image of  $L$ . Alternately, let  $\hat{K} \subset D^2 \times I$  be the  $(1, 1)$ -tangle obtained by cutting  $K$  at a point, oriented from  $\hat{K} \cap D^2 \times \{0\}$  to  $\hat{K} \cap D^2 \times \{1\}$ . If  $D$  is an oriented disk in  $S^3$  with boundary  $\gamma$ , meeting  $L$  transversely in  $n$  points, we may obtain  $I_{\gamma, K, t}(L)$  by cutting open  $L$  along  $D$  and inserting the tangle consisting of  $n$  parallel copies of  $\hat{K}$ , following the  $t$  framing. In a link diagram, a box labeled  $K, t$  in a group of parallel strands indicates  $t$ -twisted infection by  $K$  along the boundary of a disk perpendicular to those strands. To be precise, we adopt the following orientation convention: If the label  $K, t$  is written horizontally and right-side-up, then  $\hat{K}$  is oriented either from bottom to top or from left to right, depending on whether the strands meeting the box are positioned vertically or horizontally.<sup>3</sup>

Given unlinked infection curves  $\gamma_1, \gamma_2$ , the image of  $\gamma_2$  in  $I_{\gamma_1, K_1, t_1}(L \cup \gamma_2)$  is again an unknot, so we may then infect by another pair  $K_2, t_2$ . We obtain the same result if we infect along  $\gamma_2$  first and then  $\gamma_1$ . In general, given an unlink  $\gamma_1, \dots, \gamma_n$ , we may infect simultaneously along all the  $\gamma_i$ ; the result may be denoted  $I_{\gamma_1, K_1, t_1; \dots; \gamma_n, K_n, t_n}(L)$ , and the order of the tuples  $(\gamma_i, K_i, t_i)$  does not matter.

If  $P$  is a knot (or link) in the standardly embedded solid torus in  $S^3$  and  $K$  is any knot, the *t-twisted satellite of  $K$  with pattern  $P$* ,  $P(K, t)$ , is defined as  $I_{\gamma, K, t}(P)$ , where  $\gamma$  is the core of the complementary solid torus. The knot  $K$  is called the *companion*. More generally, if we have a link  $L$ , we may replace a component of  $L$  by its satellite with pattern  $P$ , working in a tubular neighborhood disjoint from the other components.

<sup>3</sup>We allow both types of notation to avoid writing labels vertically.

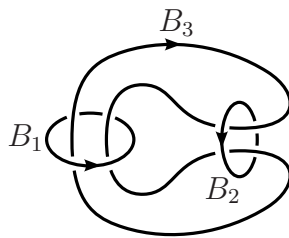
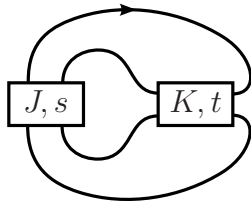


FIGURE 3. The Borromean rings.

FIGURE 4. The satellite knot  $D_{J,s}(K, t)$ .

Let  $B = B_1 \cup B_2 \cup B_3$  denote the Borromean rings in  $S^3$ , oriented as shown in Figure 3. The  $\pm t$ -twisted Whitehead double of  $K$ , is defined as

$$Wh_{\pm}(K, t) = I_{B_1, O, \mp 1; B_2, K, t}(B_3),$$

where  $O$  denotes the unknot. (Note the sign conventions: a left-handed twist in a pair of opposite strands is a positive clasp.) Moreover, we define the following generalization of Whitehead doubling: for knots  $J$  and  $K$  and integers  $s$  and  $t$ , define  $D_{J,s}(K, t)$  as the knot obtained from  $B_3$  by performing  $s$ -twisted infection by  $J$  along  $B_1$  and  $t$ -twisted infection by  $K$  along  $B_2$ :

$$D_{J,s}(K, t) = I_{B_1, J, s; B_2, K, t}(B_3).$$

(See Figure 4.) The symmetries of the Borromean rings imply:

$$\begin{aligned} D_{J,s}(K, t)^r &= D_{J^r, s}(K, t) = D_{J, s}(K^r, t) = D_{K, t}(J, s) \\ \overline{D_{J,s}(K, t)} &= D_{\bar{J}, -s}(\bar{K}, -t) \end{aligned}$$

We also introduce the convention that when the  $t$  argument is omitted, it is taken to be zero:  $D_{J,s}(K) = D_{J,s}(K, 0)$ .

The Bing double of  $K$  may be defined as  $BD(K) = I_{B_1, K, 0}(B_2 \cup B_3)$ ; we may also see this as a satellite operation where the pattern is a two-component link.

**2.2. Heegaard Floer homology and the  $\tau$  invariant.** In the 2000s, Ozsváth and Szabó [24, 25] introduced *Heegaard Floer homology*, a package of invariants for 3- and 4-dimensional manifolds that are conjecturally equivalent to earlier gauge-theoretic invariants but whose construction is much more topological in flavor. In its simplest form, given a *Heegaard diagram*  $\mathcal{H}$  for a 3-manifold  $Y$  (a certain combinatorial description of the manifold), the theory assigns a chain complex  $\widehat{CF}(\mathcal{H})$  whose chain homotopy type is independent of the choice of diagram; thus, the homology

$\widehat{\text{HF}}(Y) = H_*(\widehat{\text{CF}}(\mathcal{H}))$  is an invariant of the 3-manifold. A 4-dimensional cobordism between two 3-manifolds induces a well-defined map between their Heegaard Floer homology groups. Ozsváth and Szabó [23] and Rasmussen [26] also showed that a nulhomologous knot  $K \subset Y$  induces a filtration on the chain complex of a suitably defined Heegaard diagram, yielding an knot invariant  $\widehat{\text{HFK}}(Y, K)$  that is the  $E^1$  page of a spectral sequence converging to  $\widehat{\text{HF}}(Y)$ . For knots in  $S^3$ , the invariant  $\widehat{\text{HFK}}(S^3, K)$  categorifies the Alexander polynomial  $\Delta_K$ , and it is powerful enough to detect the unknot [22] and whether or not  $K$  is fibered [10, 20].

Furthermore, the spectral sequence from  $\widehat{\text{HFK}}(S^3, K)$  to  $\widehat{\text{HF}}(S^3) \cong \mathbb{Z}$  provides an integer-valued concordance invariant  $\tau(K)$ , which yields a lower bound on genus of smooth surfaces in the four-ball bounded by  $K$ :  $|\tau(K)| \leq g_4(K)$  [21]. In particular, any smoothly slice knot must have  $\tau(K) = 0$ . Moreover, this genus bound applies not only for surfaces in  $B^4$  but for surfaces in any rational homology 4-ball.

The  $\tau$  invariant can be used to extend many of the earlier results obstructing smooth sliceness. Hedden [12] computed the value of  $\tau$  for all twisted Whitehead doubles in terms of  $\tau$  of the original knot:

$$(1) \quad \tau(Wh_+(K, t)) = \begin{cases} 1 & t < 2\tau(K) \\ 0 & t \geq 2\tau(K). \end{cases}$$

(An analogous formula for negative Whitehead doubles follows from the fact that  $\tau(\bar{K}) = -\tau(K)$ .) In particular, if  $\tau(K) > 0$ , then  $\tau(Wh_+(K, 0)) = 1$ , so  $Wh_+(K, 0)$  (the untwisted Whitehead double of  $K$ ) is not smoothly slice. Since the  $\tau$  invariant of a strongly quasipositive knot is equal to its genus [19], Rudolph's earlier result follows from Hedden's.

The author's main theorem in [16] is a computation of  $\tau$  for all knots of the form  $D_{J,s}(K, t)$ :

**Theorem 2.1.** *Let  $J$  and  $K$  be knots, and let  $s, t \in \mathbb{Z}$ . Then*

$$\tau(D_{J,s}(K, t)) = \begin{cases} 1 & s < 2\tau(J) \text{ and } t < 2\tau(K) \\ -1 & s > 2\tau(J) \text{ and } t > 2\tau(K) \\ 0 & \text{otherwise.} \end{cases}$$

In particular, note that if  $\tau(K) > 0$  and  $s < 2\tau(J)$ , or if  $\tau(K) < 0$  and  $s > 2\tau(J)$ , then  $D_{J,s}(K, 0)$  is topologically slice (as its Alexander polynomial is 1) but not smoothly slice in any rational homology 4-ball.

The proof of Theorem 2.1 is an involved computation using the theory of *bordered Heegaard Floer homology*, developed recently by Lipshitz, Ozsváth, and Thurston [18, 17]. Briefly, the bordered theory associates to a 3-manifold with boundary a module over an algebra associated to the boundary, so that if  $Y$  is obtained by gluing together manifolds  $Y_1$  and  $Y_2$  along their common boundary, the chain complex  $\widehat{\text{CF}}(Y)$  may be computed as the derived tensor product of the invariants associated to  $Y_1$  and  $Y_2$ . If a knot  $K$  is contained in, say,  $Y_1$ , then we may obtain the filtration on  $\widehat{\text{CF}}(Y)$  corresponding to  $K$  via a filtration on the algebraic invariant of  $Y_1$ . This technique



is thus well-suited to the problem of computing Heegaard Floer invariants for knots obtained through infection operations. For more details, see [16].

**2.3. Covering link calculus.** The proof of Theorem 1.1 makes use of *covering link calculus*, first developed by Cochran and Orr [6] and used more recently by Cha and Kim [2] and others [4, 3, 30]. Let  $R$  denote any of the rings  $\mathbb{Z}$ ,  $\mathbb{Q}$ , or  $\mathbb{Z}_{(p)}$  (for  $p$  prime). A link  $L$  in an  $R$ -homology 3-sphere  $Y$  is called *topologically* (resp. *smoothly*)  $R$ -*slice* if there exists a topological (resp. smooth) 4-manifold  $X$  such that  $\partial X = Y$ ,  $H_*(X; R) = H_*(B^4; R)$ , and  $L$  bounds a locally flat (resp. smoothly embedded), disjoint union of disks in  $X$ . A link that is  $\mathbb{Z}$ -slice (in either category) is  $\mathbb{Z}_{(p)}$ -slice for all  $p$ , and a link that is  $\mathbb{Z}_{(p)}$ -slice for some  $p$  is  $\mathbb{Q}$ -slice. Also, a link in  $S^3$  that is slice (in  $B^4$ ) is clearly  $\mathbb{Z}$ -slice. The key result of Ozsváth and Szabó [21] is that the  $\tau$  invariant of any knot that is smoothly  $\mathbb{Q}$ -slice is 0.

Define two moves on links in  $\mathbb{Z}_{(p)}$ -homology spheres, called *covering moves*:

- (1) Given a link  $L \subset Y$ , consider a sublink  $L' \subset L$ .
- (2) Given a link  $L \subset Y$ , choose a component  $K$  with trivial self-linking. For any  $a \in \mathbb{N}$ , the  $p^a$ -fold cyclic branched cover of  $Y$  branched over  $K$ , denoted  $\tilde{Y}$ , is a  $\mathbb{Z}_{(p)}$ -homology sphere, and we consider the preimage  $L'$  of  $L$  in  $\tilde{Y}$ .

We say that  $L' \subset Y'$  is a  $p$ -*covering link* of  $L \subset Y$  if  $L'$  can be obtained from  $Y'$  using these moves.

The key fact is the following:

**Proposition 2.2.** *If  $L$  is (topologically or smoothly)  $\mathbb{Z}_{(p)}$ -slice, then any  $p$ -covering link of  $L$  is also (topologically or smoothly)  $\mathbb{Z}_{(p)}$ -slice.*

To prove that the second covering move preserves  $\mathbb{Z}_{(p)}$ -sliceness, we take the branched cover of the  $X$  over the slice disk for  $K$ ; the resulting 4-manifold is a  $\mathbb{Z}_{(p)}$ -homology 4-ball by a well-known argument (see, e.g., [13, page 346]). Thus, a strategy for showing a link  $L$  is not slice is to find a knot that is a covering link of  $L$  and has a non-vanishing  $\mathbb{Q}$ -sliceness obstruction, such as  $\tau$ .

Note that if  $L$  is a link in  $S^3$  whose components are unknotted, then the branched cover branched over one component is again  $S^3$ . The putative 4-manifold containing a slice disk, however, may change.

Henceforth, we restrict to the case where  $p = 2$  and omit further reference to  $p$ .

### 3. PROOF OF THEOREM 1.1

The strategy for proving the first part of Theorem 1.1 is to obtain a knot  $K'$  of the form

$$K' = D_{J_1, s_1} \circ \cdots \circ D_{J_n, s_n}(K),$$

where  $s_i < 2\tau(J_i)$  for each  $i$ , as a covering link of  $Wh_+(B_T(K))$ . If  $\tau(K) > 0$ , induction using Theorem 2.1 (which we prove below) shows that  $\tau(K') = 1$ , so  $K'$  cannot be rationally smoothly slice, so  $Wh_+(D_T(K))$  cannot be smoothly slice. A similar argument works for the second part of the theorem.

The following lemmas are inspired by Van Cott's work on the sliceness of iterated Bing doubles [30]:

**Lemma 3.1.** *Let  $L$  be a link in  $S^3$ , and suppose there is an unknotted solid torus  $U \subset S^3$  such that  $L \cap U$  consists of two components  $K_1, K_2$  embedded as follows: if  $A_1, A_2$  are the components of the untwisted Bing double of the core  $C$  of  $U$ , then  $K_1 = D_{P_k, s_k} \circ \cdots \circ D_{P_1, s_1}(A_1)$  and  $K_2 = D_{Q_l, t_l} \circ \cdots \circ D_{Q_1, t_1}(A_2)$ , for some knots  $P_1, \dots, P_k, Q_1, \dots, Q_l$  and integers  $s_1, \dots, s_k, t_1, \dots, t_l$ . Let  $L'$  be the link obtained from  $L$  by replacing  $K_1$  and  $K_2$  by the satellite knot*

$$(2) \quad C' = D_{P_k, s_k} \circ \cdots \circ D_{P_1, s_1} \circ D_{R, u}(C)$$

of  $C$ , where

$$(3) \quad (R, u) = \begin{cases} (Q_1 \# Q_1^r, 2t_1) & l = 1 \\ (D_{Q_1, t_1} \circ \cdots \circ D_{Q_{l-2}, t_{l-2}}(D_{Q_{l-1}, t_{l-1}}(Q_l \# Q_l^r, 2t_l)), 0) & l > 1. \end{cases}$$

Then  $L'$  is a covering link of  $L$ .

*Proof.* Let  $T = S^3 \setminus U$ ; then  $L \setminus (K_1 \cup K_2)$  is contained in  $T$ . Note that  $K_1$  and  $K_2$  are each unknotted, since  $D_{J, s}(O, 0) = O$  for any  $J, s$ . We may untangle  $K_2$  as in Figures 5–6. Specifically,  $L$  is shown in Figures 5(a) and (b). To obtain Figure 5(c), we pull the two strands of the companion curve for  $K_1$  through the infection region marked  $Q_1, t_1$ , and then untangle the companion curve for  $K_2$ . We then repeat this procedure to obtain Figure 5(d), and  $l - 2$  more times to obtain Figure 6.

The branched double cover of  $S^3$  branched along  $K_2$  is again  $S^3$ ; consider the preimage of  $K_1 \cup (L \cap T)$ , shown in Figure 7. (The knot orientation conventions for infections are important here, since the knots  $Q_i$  need not be reversible.) Since  $T$  is contained in a ball disjoint from  $K_1$ , the sublink  $L \cap T$  lifts to two identical copies, each contained in a solid torus. The preimage of  $K_2$  also consists of two components, and each is the  $D_{P_k, s_k} \circ \cdots \circ D_{P_1, s_1}$  satellite of the companion curve shown. A sublink consisting of one lift of each component (either the blue or the black part of Figure 7) is redrawn in Figure 8(a) in the case where  $l = 1$  and in Figure 8(b) in the case where  $l > 1$ . In the former case, the companion curve shown is  $D_{Q_l \# Q_l^r, 2t_l}(C)$ , where  $C$  is the core circle of the complement of  $T$ . In the latter case, it is  $D_{R, 0}(C)$ , where we obtain  $R$  by connecting the ends of one of the two parallel strands that pass through the red box in Figure 8(b). (A local computation shows that the linking number of these two strands is zero, so  $D_{R, 0}$  is the correct operator.) The knot  $R$ , shown in Figure 9, is then identified as

$$D_{Q_1, t_1} \circ \cdots \circ D_{Q_{l-2}, t_{l-2}}(D_{Q_{l-1}, t_{l-1}}(Q_l \# Q_l^r, 2t_l)). \quad \square$$

**Lemma 3.2.** *Let  $C$  be a knot, let  $U$  be a regular neighborhood of  $C$ , and let  $A_1, A_2 \subset U$  be the components of  $BD(C)$ . Let  $K_1 = D_{P_k, s_k} \circ \cdots \circ D_{P_1, s_1}(A_1)$  and  $K_2 = D_{Q_l, t_l} \circ \cdots \circ D_{Q_1, t_1}(A_2)$ , for some knots  $P_1, \dots, P_k, Q_1, \dots, Q_l$  and integers  $s_1, \dots, s_k, t_1, \dots, t_l$ . Let  $C'$  be the knot defined by (2) and (3). Then  $C'$  is a covering link of  $K_1 \cup K_2$ .*

*Proof.* The proof is almost identical to that of Lemma 3.1. The only difference is that  $S^3 \setminus U$  is now a knot complement rather than a solid torus containing some additional link components. The double branched cover over  $K_2$  contains consists of the complement of the two solid tori shown in Figure 7, glued to two copies of  $S^3 \setminus U$ ,



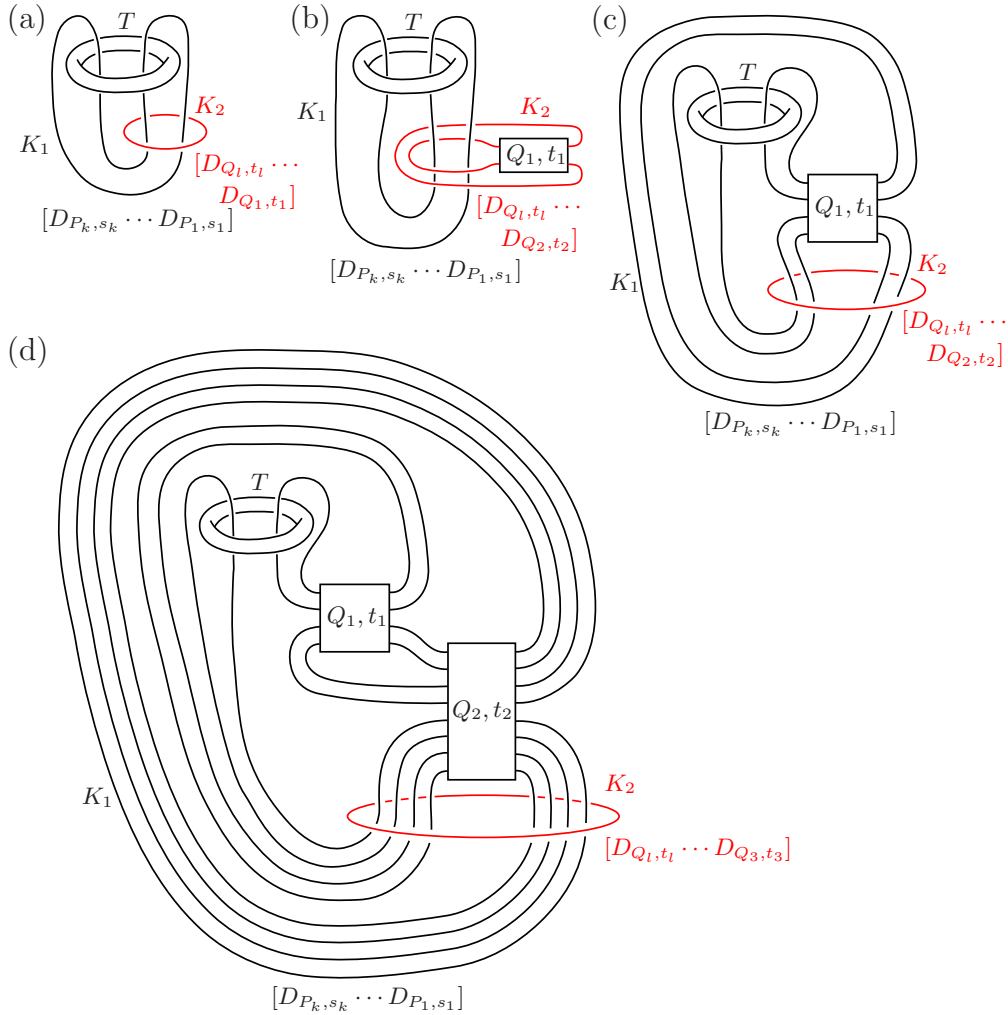


FIGURE 5. The link described in Lemma 3.1. All but the two components shown are contained in the interior of the solid torus  $T$ . We denote a satellite knot by writing the pattern in brackets near the companion curve; thus, for instance,  $K_1 = D_{P_k, s_k} \circ \cdots \circ D_{P_1, s_1}(A_1)$ , where  $A_1$  is the curve shown.

gluing Seifert-framed longitude to meridian and vice versa. The resulting manifold is again  $S^3$ , however. The rest of the proof proceeds *mutatis mutandis*. (Alternately, we may simply replace each of the solid tori in Figures 5–9 by a box marked  $C, 0$ , and proceed as before.)  $\square$

A *labeled binary tree* is a binary tree with each leaf labeled with a satellite operation. Given a knot  $K$  and binary tree  $\mathcal{T}$  with underlying tree  $T$ , let  $S_{\mathcal{T}}(K)$  be the link obtained from  $B_T(K)$  by replacing each component with the satellite specified by the label of the corresponding leaf. If  $\mathcal{T}$  has two adjacent leaves labeled  $D_{P_k, s_k} \circ \cdots \circ D_{P_1, s_1}$  and  $D_{Q_1, t_1} \circ \cdots \circ D_{Q_1, t_1}$ , form a new labeled tree  $\mathcal{T}'$  by deleting these two leaves and labeling the new leaf either  $D_{P_k, s_k} \circ \cdots \circ D_{P_1, s_1} \circ D_{Q_1 \# Q_1^{r, 2t_1}}(C)$  or

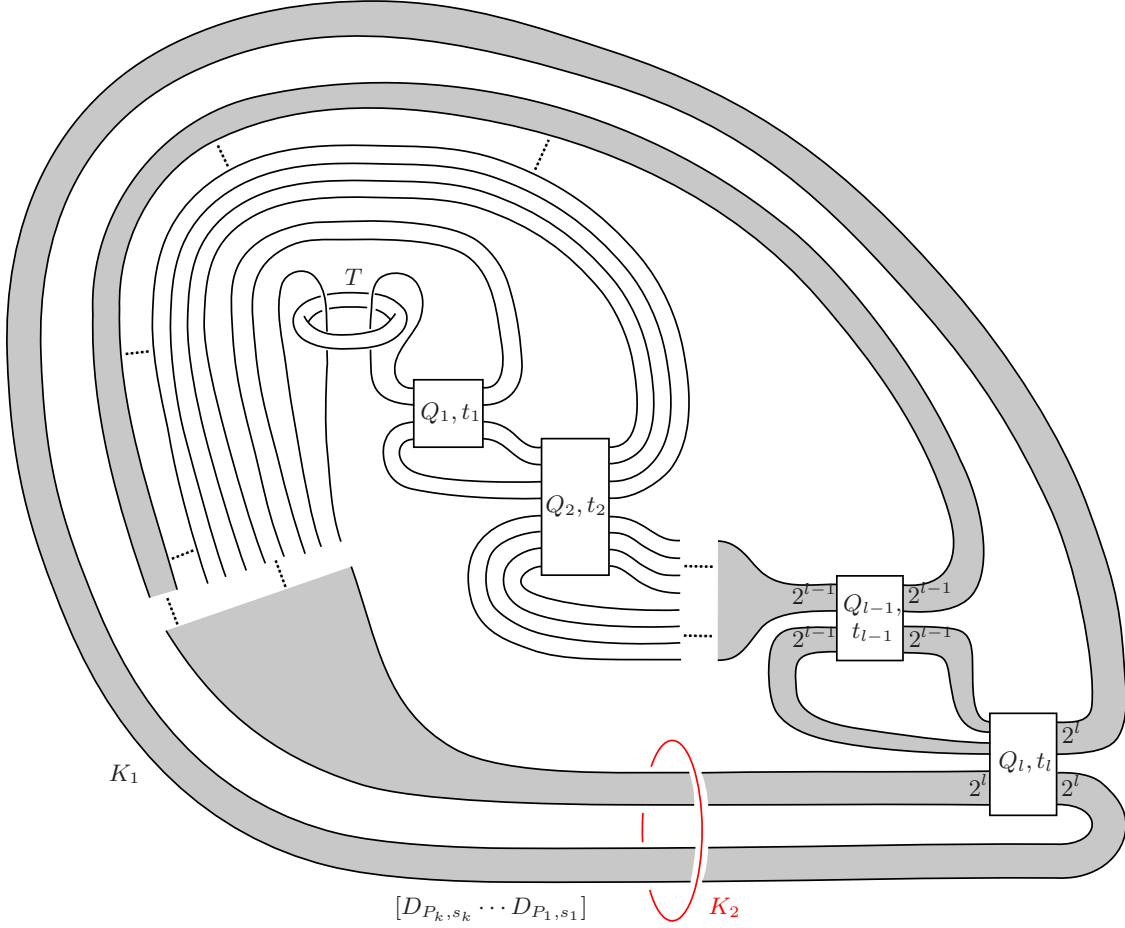


FIGURE 6. The link described in Lemma 3.1, after isotopies. A shaded region with a number represents that many parallel strands.

$D_{P_k, s_k} \circ \cdots \circ D_{P_1, s_1} \circ D_{R, 0}$ , according to whether  $l = 1$  or  $l > 1$ , respectively, where,  $R = D_{Q_1, t_1} \circ \cdots \circ D_{Q_{l-2}, t_{l-2}}(D_{Q_{l-1}, t_{l-1}}(Q_l \# Q_l^r, 2t_1))$  in the latter case. We call this move a *collapse*. Lemmas 3.1 and 3.2 then say that  $S_{\mathcal{T}'}(K)$  is a covering link of  $S_{\mathcal{T}}(K)$ .

Theorem 2.1 and equations (2) and (3), along with the additivity of  $\tau$  under connect sum, imply:

**Proposition 3.3.** *Suppose  $\mathcal{T}'$  is obtained from  $\mathcal{T}$  by collapsing leaves labeled  $D_{P_k, s_k} \circ \cdots \circ D_{P_1, s_1}$  and  $D_{Q_l, t_l} \circ \cdots \circ D_{Q_1, t_1}$ , where  $s_i < 2\tau(P_i)$  and  $t_i < 2\tau(Q_i)$  for all  $i$ . Then the label of the new leaf of  $\mathcal{T}'$  has the form  $D_{R_{k+1}, u_{k+1}} \circ \cdots \circ D_{R_1, u_1}$ , where  $u_i < 2\tau(R_i)$ .  $\square$*

*Proof of Theorem 1.1.* For the first part of the theorem, note that in the new notation,  $Wh_+(B_T(K)) = S_{\mathcal{T}}(K)$ , where every leaf of  $\mathcal{T}$  is labeled  $D_{O, -1}$ . Every label in  $\mathcal{T}$  satisfies the hypotheses of Proposition 3.3. Using this proposition, we inductively collapse every pair of leaves of  $\mathcal{T}$  until we have a single vertex labeled  $D_{P_k, s_k} \circ \cdots \circ$

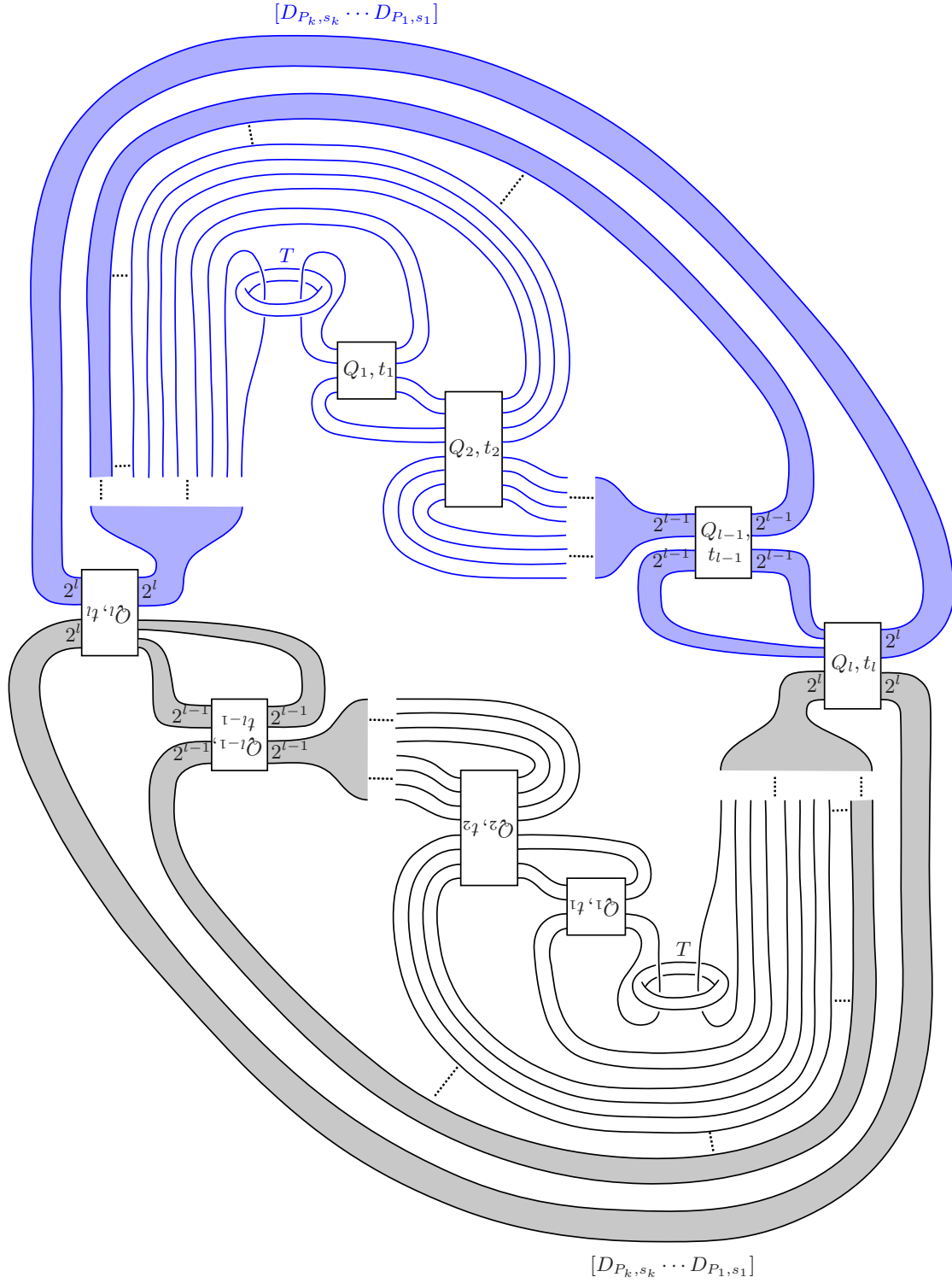


FIGURE 7. The preimage of the link in Figure 6 in the double-branched cover of  $S^3$  over  $K_2$  (shown without the upstairs branch set).

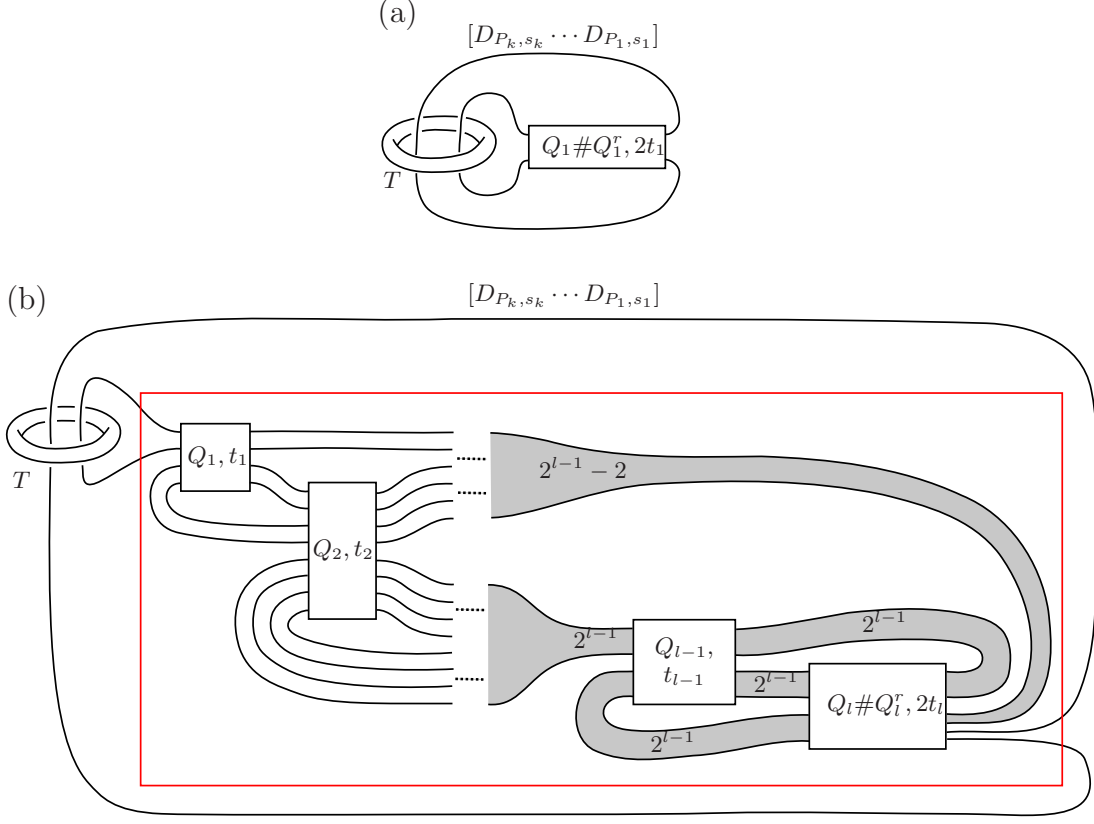


FIGURE 8. The sublink shown in blue in Figure 7 is the  $D_{P_k, s_k} \circ \cdots \circ D_{P_1, s_1}$  satellite of: (a) when  $l = 1$ ,  $D_{Q_1 \# Q_1^r, 2t_1}(C)$ ; (b) when  $l > 1$ ,  $D_{R, 0}(C)$ , where  $R$  is the knot in Figure 9.

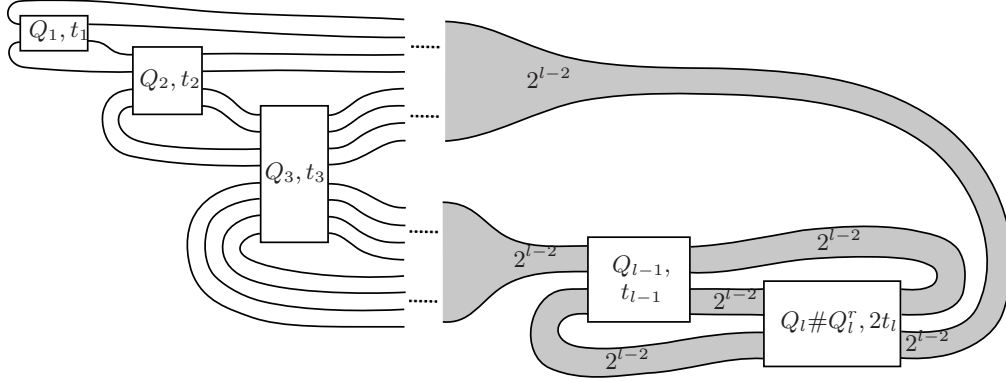


FIGURE 9. The knot  $R$  in the proof of Lemma 3.1.

$D_{P_1, s_1}$ , for knots  $P_1, \dots, P_k$  and integers  $s_1, \dots, s_k$  with  $s_i < 2\tau(P_i)$ . Thus, the knot  $D_{P_k, s_k} \circ \cdots \circ D_{P_1, s_1}(K)$  is a covering link of  $Wh_+(B_T(K))$ . By Theorem 2.1,  $\tau(D_{P_k, s_k} \circ \cdots \circ D_{P_1, s_1}(K)) = 1$ . Thus,  $D_{P_k, s_k} \circ \cdots \circ D_{P_1, s_1}(K)$  cannot be smoothly slice in a rational homology 4-ball, so  $Wh_+(B_T(K))$  cannot be smoothly slice.

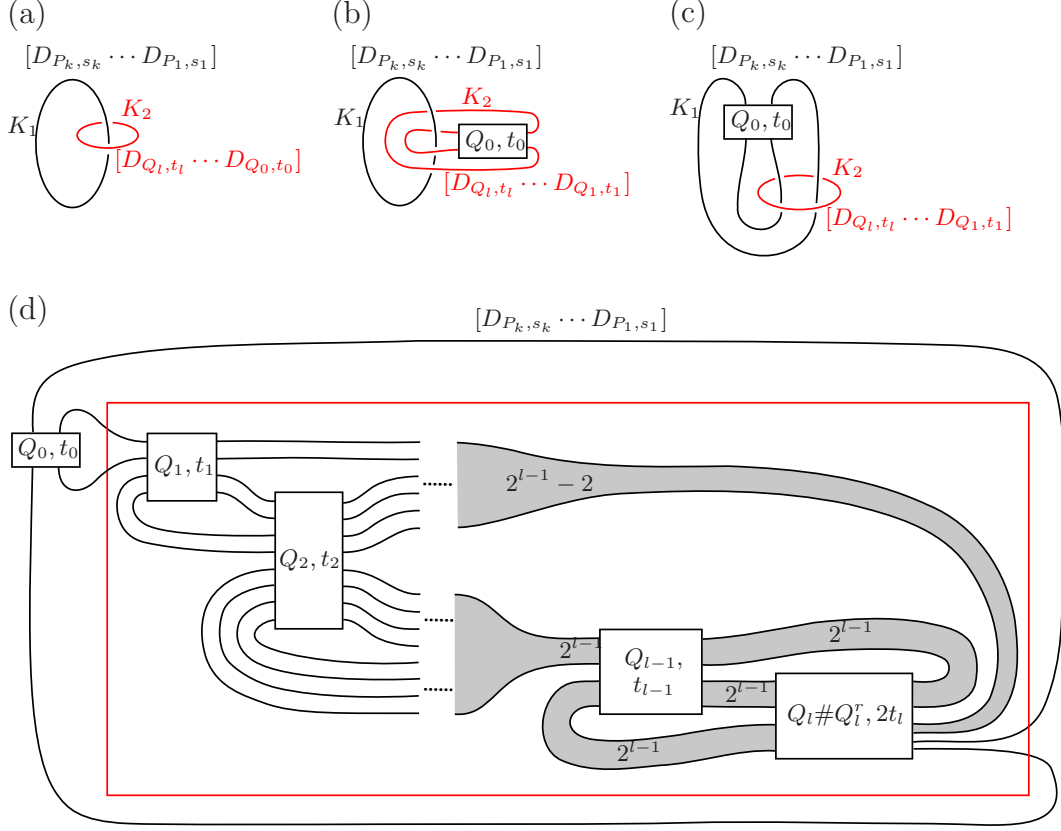


FIGURE 10. The proof of the second part of Theorem 1.1.

For the second part, the same argument as above shows that by using covering moves, we may replace  $Wh_+(B_{T_1}(K_1) \cup B_{T_2}(K_2))$  with a two-component link of the form

$$D_{P_k, s_k} \circ \cdots \circ D_{P_1, s_1}(K_1) \cup D_{Q_l, t_l} \circ \cdots \circ D_{Q_0, t_0}(K_2),$$

shown in Figure 10(a), where  $s_i < 2\tau(P_i)$  and  $t_i < 2\tau(Q_i)$  for all  $i$ . (We start with  $Q_0$  and  $t_0$  for notational reasons.) After the isotopies in Figure 10(a–c), note the similarity to Figure 5. We may thus proceed just as in the proof of Lemma 3.1, with suitable modifications to Figures 6–8, to obtain the knot shown in Figure 10(d) as a covering link of  $Wh_+(B_{T_1}(K_1) \cup B_{T_2}(K_2))$ . This knot is

$$D_{P_k, s_k} \circ \cdots \circ D_{P_1, s_1}(D_{R, u}(Q_0, t_0)),$$

where  $(R, u)$  is as in (3). This knot has  $\tau = 1$  by Theorem 2.1, completing the proof.  $\square$

#### 4. STRONGLY QUASIPOSITIVE KNOTS AND SLICENESS

We conclude with a brief discussion of strongly quasipositive knots, which played a role in an earlier version of this paper.

A knot or link  $L$  is called *quasipositive* if it is the closure of a braid that is the product of conjugates of the standard positive braid generators  $\sigma_i$  (but not their inverses). It is called *strongly quasipositive* if it is the closure of a braid that is the product of words of the form  $\sigma_i \dots \sigma_{j-1} \sigma_j \sigma_{j-1}^{-1} \dots \sigma_i^{-1}$  for  $i < j$ . A strongly quasipositive link naturally admits a particular type of Seifert surface determined by this braid form, and an embedded surface in  $S^3$  is called *quasipositive* if it is isotopic to such a surface. In other words, a link is strongly quasipositive if and only if it bounds a quasipositive Seifert surface.

A link  $L$  is quasipositive if and only if it is a *transverse  $\mathbb{C}$ -link*: the transverse intersection of  $S^3 \subset \mathbb{C}^2$  with a complex curve  $V$ . If  $L$  is strongly quasipositive, then the Seifert surface determined by the braid form is isotopic to  $V \cap B^4$ .

For a knot  $K$  and  $t \in \mathbb{Z}$ , let  $A(K, t)$  be an annulus in  $S^3$  whose core circle is  $K$  and whose two boundary components are  $t$ -framed longitudes of the core. Given two unlinked annuli  $A$  and  $A'$ , let  $A * A'$  denote the surface obtained by plumbing  $A$  and  $A'$  together. (To be precise, we must orient the core circles of  $A$  and  $A'$  and specify the sign of their intersection in  $A * A'$ .)

The following is a summary of some of Rudolph's results [27, 28, 29] on strongly quasipositive knots:

**Theorem 4.1.**

- (1) If  $K$  is a strongly quasipositive knot other than the unknot, then  $K$  is not smoothly slice.
- (2) A knot  $K$  is strongly quasipositive if and only if  $A(K, 0)$  is a quasipositive surface.
- (3) If  $K$  and  $K'$  are strongly quasipositive, then  $K \# K'$  is strongly quasipositive.
- (4) The annulus  $A(K, t)$  is quasipositive if and only if  $t \leq TB(K)$ , where  $TB(K)$  denotes the maximal Thurston–Bennequin number of  $K$ .
- (5) If  $A$  and  $A'$  are annuli, then the surface  $A * A'$  is quasipositive if and only if  $A$  and  $A'$  are both quasipositive.

Rudolph's original proof of (1) relies on the fact that complex curves are genus-minimizing, a major theorem proven by Kronheimer and Mrowka [15] using gauge theory. Since a strongly quasipositive knot  $K$  has a Seifert surface that is isomorphic to a complex curve, we thus see that  $g_4(K) = g(K)$ ; in particular, if  $K$  is nontrivial, then  $g_4(K) > 0$ . Subsequently, Livingston [19] proved that both of these genera are equal to  $\tau(K)$  when  $K$  is strongly quasipositive. (For more on the relationship between  $\tau$  and quasipositivity, see Hedden [11].)

The untwisted  $\pm$  Whitehead double of  $K$ ,  $Wh_{\pm}(K)$ , is the boundary of  $A(K, 0) * A(O, \mp 1)$ , where  $O$  denotes the unknot. Thus, Theorem 4.1 implies that if  $K$  is strongly quasipositive and nontrivial, then  $Wh_+(K)$  is strongly quasipositive and nontrivial, hence not smoothly slice. More generally, the plumbing  $A(J, s) * A(K, t)$  is a Seifert surface for  $D_{J,s}(K, t)$ , so if  $J$  and  $K$  are strongly quasipositive and  $s, t \leq 0$ , then  $D_{J,s}(K, t)$  is strongly quasipositive. Moreover, if neither of the pairs  $(J, s)$  and  $(K, t)$  equals  $(O, 0)$ , then  $D_{J,s}(K, t)$  is nontrivial, hence not smoothly slice. Furthermore, in this case  $\tau(D_{J,s}(K, t)) = 1$  since the  $\tau$  invariant of a strongly quasipositive knot is



equal to its genus by a result of Livingston [19]. Using this observation, we may prove a weakened version of Theorem 1.1 in which the knot  $K$  is assumed to be strongly quasipositive without ever making reference to Theorem 2.1.

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